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Semiclassical limit of chaotic eigenfunctions

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Abstract
A generic chaotic eigenfunction has a non-universal contribution consisting of scars of short periodic orbits. This contribution, which cannot be predicted by a model of random universal waves, survives the semiclassical limit (when $\hbar$ goes to zero). In this limit, the sum of scarred intensities only depends on $\eta \equiv (f-1)\left(\sum \lambda_i^2\right)^{1/2} / h_T$, with f the degrees of freedom, $\{\lambda_i\}$ the set of positive Lyapunov exponents and h_T the topological entropy. Moreover, taking into account that relative fluctuations of the scarred intensities tend to zero as $1/|\ln \hbar|$, we are able to provide a detailed description of a generic chaotic eigenfunction in the semiclassical limit. Our conclusions were verified in the Bunimovich stadium billiard.

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1. Introduction

Berry [1] and Voros [2] proposed a semiclassical description of chaotic eigenfunctions by considering the surface of constant energy as the unique classical invariant able to support them. This guess should be right if the time required for the definition of individual eigenfunctions was infinite. However, the required Heisenberg time $T_{\rm H}$ is finite for finite values of $\hbar$, even though it tends to infinity in the semiclassical limit (SL).

It is worth emphasizing that fluctuations in the Berry–Voros description tend to infinity in the SL (when $\hbar$ goes to zero). But this statement demands an explanation because in the literature it is usual to find expressions like '*. . . this description is supported by the Shnirelman theorem*'. For instance, the peaks of a Husimi function are distributed in phase space according to the classical ergodic measure, being the heights of the peaks order $|\ln \hbar|$ with respect to this measure [3]. Nevertheless, taking into account that the width of these peaks is of order $\sqrt{\hbar}$, a classical smoothing[1] is sufficient for avoiding out fluctuations in the SL. In this context, the Shnirelman theorem [4] can be expressed as follows: *a classical smoothing is sufficient for the elimination of fluctuations of generic chaotic eigenfunctions in the* SL.

It is clear that in order to study the SL of individual chaotic eigenfunctions, it is necessary to deal with fluctuations. Therefore, the following question arises: are fluctuations independent of the used representation? The Husimi function describes fluctuations on the basis of coherent states. This basis is universal in the sense that it does not contain dynamical information of the system and consequently, it can be used in all systems. On the other hand, a basis with dynamical information is evidently non-universal; for instance, a basis with information of the structure of periodic orbits (POs). This basis is certainly motivated by a lot of previous works about the structure of chaotic eigenfunctions.

In 1983, the influence of short POs on the structure of some chaotic eigenfunctions of the Bunimovich stadium billiard was noted [5]. In 1984, Heller [6] demonstrated that the *shortest* POs are able to leave scars on some chaotic states. After that, scars were considered within the framework of the periodic orbit theory by making an energy average over a large number of eigenfunctions [7]; this approach provides information about mean properties of the so-called scar phenomenon but eliminates fluctuations. Then, resumation techniques have been applied in order to deal with individual eigenfunctions [8]; nevertheless, the method requires the computation of an enormous number of POs (which increases exponentially with T_{H}), preventing a transparent, simple and detailed description of the phenomenon.

Recently, we have observed in the stadium billiard that a basis of wavefunctions living along short POs is naturally embedded in the set of eigenfunctions [9]. Based on that observation, a semiclassical theory of short POs was proposed [10, 11], where the number of required orbits increases at most linearly with T_{H}. In this framework, an eigenfunction is represented by a superposition of wavefunctions living in the neighbourhood of short POs, the so-called scar functions in [12]. Then, the essential idea is to replace long POs, used in the standard periodic orbit theory, by an interaction between pairs of short POs. By interaction we mean Hamiltonian matrix elements between scar functions.

Our aim in this paper is to study the SL of chaotic eigenfunctions in the representation of scar functions, and its comparison with the corresponding description in terms of universal bases. Starting with the mean properties of a scar function over a large number of eigenfunctions, we study its action on individual eigenfunctions within a statistical model. The interaction is indirectly included through its action on the set of amplitudes on the basis of scar functions; specifically, amplitudes will be random variables compatible with the mean properties and the interaction. Then, we evaluate the highest intensities (the square modulus of the amplitudes) which cannot be predicted by a universal description. As a result of this analysis, we conclude that the sum of such intensities survives the SL. Finally, we check our conclusions in the Bunimovich stadium billiard.

2. A statistical connection between the scar function basis and the set of eigenfunctions

Scar functions are the objects on which we focus our attention. A scar function is a wavefunction which uses dynamical information in the vicinity of a PO up to the Ehrenfest time; its semiclassical construction is explained in [12]. Let γ be a PO of the system with period T_γ and the Lyapunov exponent λ_γ, and let ϕ_γ be the corresponding scar function with

[1] By classical smoothing we refer, for instance, to a Gaussian smoothing where the dispersion of the used Gaussian function is of the order $\hbar^0$.

Bohr–Sommerfeld (BS) energy E_γ.[2] Moreover, let $\{\varphi_\mu\}$ be the set of normalized eigenfunctions of the system with eigenenergies E_μ. Mean properties of ϕ_γ in the spectrum are characterized, through the intensities $I_\mu \equiv |\langle\varphi_\mu|\phi_\gamma\rangle|^2$, by the following relations [12]:

$$\sum_\mu I_\mu = 1 + \mathrm{O}(\hbar) \qquad \sum_\mu E_\mu I_\mu = E_\gamma + \mathrm{O}(\hbar)$$

and

$$\sigma_\gamma \equiv \sqrt{\sum\nolimits_\mu (E_\mu - E_\gamma)^2 I_\mu} = \frac{\Gamma\hbar\lambda_\gamma}{2} + \mathrm{O}(\hbar^2) \tag{1}$$

where $\Gamma = 2\pi/|\ln\hbar| + \mathrm{O}(|\ln\hbar|^{-2})$. This means that ϕ_γ is highly localized in a range order σ_γ around E_γ.

Equation (1) only works for Hamiltonian systems with two degrees of freedom, but the generalization to f degrees of freedom is simply obtained in leading order by substituting λ_γ for $\left(\sum \lambda_{\gamma,i}^2\right)^{1/2}$, where $\{\lambda_{\gamma,i}\}$ is the set of $f-1$ positive Lyapunov exponents of γ.[3] With this substitution, the leading order of the energy dispersion, where $1/|\ln\hbar|$ is the small parameter, results

$$\sigma_\gamma \simeq \frac{\pi\hbar}{|\ln\hbar|}\left(\sum \lambda_{\gamma,i}^2\right)^{1/2}. \tag{2}$$

We emphasize that scar functions are constructed by requiring minimum energy dispersion; in this respect, equation (2) provides the minimum dispersion compatible with a structure living in the vicinity of γ. The most striking characteristic of σ_γ is the logarithmic dependence on $\hbar$ that reveals, in fact, the relation of the scar function with the Ehrenfest time.

A detailed numerical analysis in the stadium billiard [14] shows that the mean value of the intensities, in a range order $\hbar^{3/2}$ [4] around the energy E, is given by a Gaussian function of $E - E_\gamma$. Moreover, the intensities fluctuate around the mean value following a chi-square distribution with one degree of freedom [15]. We believe that the Gaussian behaviour is related to the mixing property of chaotic systems [13] and consequently of general validity. For this reason we propose the following expression for the mean value of the intensities as a function of E:

$$\overline{I}(E) = \frac{\exp\left[-(E-E_\gamma)^2\big/2\sigma_\gamma^2\right]}{\sqrt{2\pi}n} \tag{3}$$

where

$$n \equiv \sigma_\gamma \rho_E = \mathrm{O}(\hbar^{1-f}/|\ln\hbar|) \tag{4}$$

is the number of eigenenergies contained in one energy dispersion range, and $\rho_E = \mathrm{O}(\hbar^{-f})$ is the energy density[5]. Furthermore, fluctuations being one of the quantum manifestations of chaos, we hope that intensities of general chaotic systems also fluctuate following a chi-square distribution. Below, we incorporate this guess through a random wave hypothesis.

[2] The BS energies of a PO satisfy the relation $S/\hbar - \mu\pi/2 = 2n\pi$, where n is an integer, S is the dynamical action along the PO and μ is the Maslov index.

[3] After a canonical transformation, the transverse motion can be locally described by the hyperbolic Hamiltonian $\sum \lambda_{\gamma,i} p_i q_i$. Then, each conjugated plane $p_i q_i$ provides an independent contribution to σ_γ^2, given in leading order by $(\pi\hbar\lambda_{\gamma,i}/|\ln\hbar|)^2$.

[4] This range contains a large number of eigenenergies but at the same time it is much smaller than σ_γ.

[5] Note that $\overline{I}(E)$ is normalized by the relation $\int \overline{I}(E)\rho_E \, \mathrm{d}E = 1$.

2.1. Scar functions in the basis of eigenfunctions

In what follows, we study the intensities of ϕ_γ within a statistical model, being the main purpose to estimate their mean values and dispersions, and to provide a range in the spectrum where they can be found. The implicit assumption is that $\phi_\gamma = \sum \langle \varphi_\mu | \phi_\gamma \rangle \varphi_\mu$ looks like a random wave; that is, amplitudes are independent Gaussian distributions with zero means, satisfying global properties imposed by equation (3). Moreover, as we will develop a formalism of individual intensities in terms of the small parameter $1/|\ln \hbar|$, corrections depending algebraically on $\hbar$ result irrelevant[6]. We start by noting that an intensity is characterized by its height I and its position E in the spectrum. With the change of variable $\epsilon \equiv (E - E_\gamma)\rho_E$, equation (3) takes the form

$$\overline{I}(\epsilon) = \frac{\exp[-\epsilon^2/2n^2]}{\sqrt{2\pi} n}. \tag{5}$$

In this way, the statistical model only depends on the parameter n. The position satisfies a uniform distribution in the range $-N/2 < \epsilon < N/2$, where N is the number of intensities of ϕ_γ; the precise value of N is irrelevant and we only require that $N > n|\ln \hbar|$. Moreover, the height I at a given position ϵ satisfies a chi-squared distribution with β degrees of freedom and mean value $\overline{I}(\epsilon)$; for systems with (without) time reversal symmetry, β is equal to 1 (2). Then, an individual intensity is described by the following probability density:

$$p(I, \epsilon) = \frac{\chi[I/\overline{I}(\epsilon)]}{N\overline{I}(\epsilon)} \tag{6}$$

where χ is a chi-square probability density with β degrees of freedom and mean value unity[7].

We would like to evaluate, in first place, the heights of the highest intensities independent of their position in the spectrum. To do this simply, we integrate equation (6) on the random variable ϵ

$$p(I) = \int_{-N/2}^{N/2} p(I, \epsilon)\, \mathrm{d}\epsilon.$$

Then, the set of intensities of ϕ_γ is taken into account by N independent random variables with common probability density $p(I)$.[8] Let x_1 be the greatest of the intensities, x_2 be the second one and so on; that is, we arrange the set of intensities in an order of magnitude. Following the general method described in chapter 28 of [16], we obtain expressions for the mean value and dispersion of x_j [9]

$$\overline{x_j} = \sqrt{\frac{2}{\pi}} \frac{\ln(n/j)}{\beta n} \left[1 + \mathrm{O}\left(\frac{1}{\ln(n/j)}\right)\right] \tag{7}$$

$$\sigma_{x_j} = \sqrt{\frac{2}{\pi}} \frac{a_j}{\beta n} \left[1 + \mathrm{O}\left(\frac{1}{\ln(n/j)}\right)\right] \tag{8}$$

[6] For this reason, even though the energy density in a range order σ_γ has relative variations order $\hbar$, we simply take the constant value ρ_E.

[7] $\chi(x) = \Gamma(\beta/2)^{-1}(\beta x/2)^{\beta/2-1}(\beta/2)e^{-\beta x/2}$, with $\Gamma(\,)$ the Gamma function.

[8] N independent random variables with common probability density $p(I, \epsilon)$ provide a Poisson energy level spacing distribution, being fluctuations $\mathrm{O}(N^{-1/2}) = \mathrm{O}(\hbar^{(f-1)/2})$. However, fluctuations are $\mathrm{O}(\hbar^f)$ as a consequence of the strong rigidity of chaotic spectra. In this respect, the integration along the energy axis eliminates fluctuations, providing a better description of the actual situation. Any way, differences among the previous possibilities depend algebraically on $\hbar$; so, they are all equivalent in our model.

[9] Actually, for $\beta = 1$, the error in equation (7) is of the order $[\ln\ln(n/j)]/\ln(n/j)$.

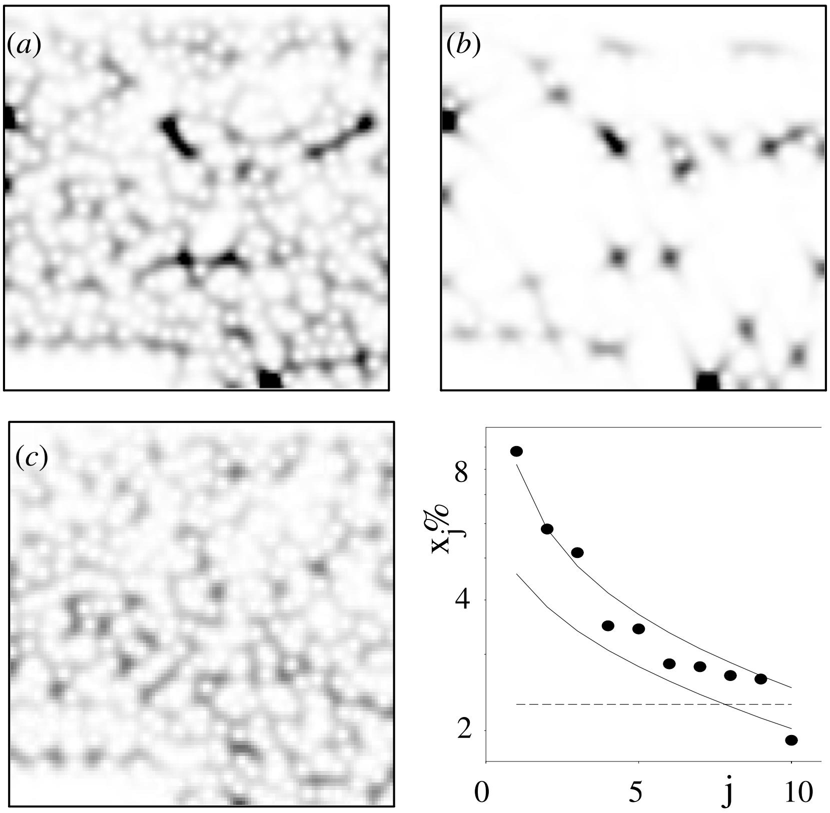

Figure 1. Linear Husimi density plots of (*a*) the state number 141 755 of the desymmetrized stadium billiard with radius 1 and area $1+\pi/4$, (*b*) its non-universal contribution consisting of nine scar functions, and (*c*) its universal contribution given by 819 plane waves. Finally, it is displayed the set of scarred intensities (dots) and the theoretical curves $\overline{x_j} \pm \sigma_{x_j}$; horizontal line shows the value η/n of the highest universal intensity.

with

$$a_j \equiv \left(\frac{\pi^2}{6} - \sum_{i=1}^{j-1} \frac{1}{i^2}\right)^{1/2} = \frac{1 + 1/4j + \mathrm{O}(j^{-2})}{\sqrt{j}}.$$

Precise expressions for $\overline{x_j}$, with the remainder $\mathrm{O}(\ln(n/j)^{-4})$, and σ_{x_j} can be found in [14]. These expressions are required for checking experimental data because, for instance, $\ln n \simeq 3.8$ in the very high energy region of figure 1. With respect to equation (7), we point out that (i) the dependence on the ratio n/j is a consequence of the arrangement of the intensities, (ii) the logarithmic dependence and the factor β are related to the asymptotically exponential behaviour of χ, (iii) the factor $\sqrt{2/\pi}$ depends on the selected Gaussian function in equation (3) and (iv) the factor n normalizes the sum of intensities to unity. Moreover, it is evident from equations (7) and (8) that relative fluctuations of the intensities tend to zero in the SL

$$\sigma_{x_j}/\overline{x_j} \sim 1/\ln n \sim 1/|\ln \hbar|. \tag{9}$$

This result is a consequence of the arrangement of the intensities and the asymptotically exponential behaviour of χ.

Second, we would like to know where the intensity x_j can be found in the spectrum. From equation (9), $x_j \simeq \overline{x_j}$ in leading order, so the probability density of finding x_j near ϵ is given (in leading order) by

$$p_j(\epsilon) \simeq \frac{p(\overline{x_j}, \epsilon)}{\int_{-N/2}^{N/2} p(\overline{x_j}, \epsilon')\, \mathrm{d}\epsilon'}.$$

A straightforward calculation of this equation shows that the position E_j of x_j is described by a distribution (Gaussian in the SL) with mean value E_γ and dispersion

$$\sigma_{E_j} = \frac{\sigma_\gamma}{\sqrt{\ln(n/j)}}[1 + \mathrm{O}(1/\ln n)]. \tag{10}$$

This means that the highest intensities of ϕ_γ tend to E_γ quicker than σ_γ goes to zero. Moreover, $\sigma_{E_j}/\Delta E_\gamma$ tends to zero as $|\ln\hbar|^{-3/2}$, where $\Delta E_\gamma = 2\pi\hbar/T_\gamma$ is the distance between consecutive BS energies of γ. Below, we will see that x_j is always a scar for sufficiently small values of $\hbar$. Accepting this fact, equation (10) imposes a necessary condition of the BS quantization type for scarring; that is, *scars of γ accumulate in the vicinity of E_γ* in the SL.

2.2. Eigenfunctions in the representation of scar functions

So far, we have described scar functions on the basis of eigenfunctions. Now, we show that the representation of eigenfunctions on the basis of scar functions has essentially the same description as in the SL. Let E be an energy living in the extreme semiclassical region; this means that $|\ln\hbar| \gg 1$ when the energy is measured in units of E and the time in units of h_T^{-1}, where h_T is the topological entropy[10]. Let $\{\phi_\gamma\}$ be the set of scar functions of short POs with BS energies near E (in appendix B, the concept of short PO is precisely established). This set defines a very distinctive basis of wavefunctions in order to describe eigenfunctions in the vicinity of E. First, the elements of this basis are strongly localized in energy (see equation (2)). Second, this basis is quasi-orthogonal. The maximum overlap between scar functions is of the order $1/|\ln\hbar|$, but most of the overlaps are of the order $\hbar^{1/2}$;[11] in fact, the fraction of overlaps of the order $1/|\ln\hbar|$ goes to zero in the SL. Third, the density of BS energies is equal to ρ_E because we incorporate POs up to this condition is satisfied [10].

From the second and third properties we arrive at the conclusion that the set of scar functions is a complete basis. Moreover, by quasi-orthogonality

$$\varphi_\mu = \sum c_{\mu,\gamma}\phi_\gamma \tag{11}$$

with

$$c_{\mu,\gamma} = \langle\phi_\gamma|\varphi_\mu\rangle[1 + \mathrm{O}(\hbar^{1/2})] \tag{12}$$

and where the number of relevant terms is of the order $\hbar^{1-f}/|\ln\hbar|$ in accordance with equation (4). Therefore, the same statistical description works by using

$$\sigma_\mu^2 \equiv \sum_\gamma |c_{\mu,\gamma}|^2 (E_\mu - E_\gamma)^2$$

in place of σ_γ^2.

[10] The number of primitive POs with period less that T is given by $\sim\exp(Th_T)/Th_T$. So, h_T^{-1} is of the order of the shortest periods.

[11] This result is derived in [13] for the $f = 2$ case. We believe that for higher degrees of freedom the overlap goes quickly to zero, possibly as $\hbar^{(f-1)/2}$. However, as the precise dependence with $\hbar$ is not relevant to our discussion, we retain the upper bound $\hbar^{1/2}$.

We can obtain an estimation of σ_μ from the following relations:

$$\sigma_\mu^2 = \langle\sigma_\mu^2\rangle[1 + \mathrm{O}(\hbar^{(f-1)/2})]$$
$$\langle\sigma_\mu^2\rangle = \langle\sigma_\gamma^2\rangle[1 + \mathrm{O}(\hbar^{1/2})]$$
$$\left\langle \sum_i \lambda_{\gamma,i}^2 \right\rangle = \sum_i \lambda_i^2 + \mathrm{O}(1/|\ln\hbar|)$$

where $\{\lambda_i\}$ is the set of $f-1$ positive Lyapunov exponents of the system. The first of these equations means that fluctuations of σ_μ^2 are very small, being its mean value a good estimation; this result is simply obtained by assuming that equation (11) looks like a random wave. The second equation is a direct consequence of (12). With respect to the third equation, we note that according to the exponential proliferation of POs with the period, the mean value on the lhs is dominated by POs with the longest periods. Then, as the dispersion of $\lambda_{\gamma,i}$ around the mean value λ_i is of the order $T_\gamma^{-1/2}$, where $T_\gamma = \mathrm{O}(|\ln\hbar|)$ for the longest periods, the third equation results.

Using the previous observations and equation (2), the typical dispersion of eigenfunctions on the basis of scar functions is given by

$$\sigma \simeq \sqrt{\langle\sigma_\gamma^2\rangle} \simeq \frac{\pi\hbar}{|\ln\hbar|}\left(\left\langle \sum_i \lambda_{\gamma,i}^2 \right\rangle\right)^{1/2} \simeq \frac{\pi\hbar}{|\ln\hbar|}\left(\sum_i \lambda_i^2\right)^{1/2}. \tag{13}$$

In conclusion, using σ in place of σ_γ, equation (4) provides the relevant number of scar functions contributing to each eigenfunction. The intensities of an eigenfunction on the basis of scar functions have mean value and dispersion given by equations (7) and (8), and those POs with probability of having intensity x_j are limited by equation (10).

We have used a random wave hypothesis to connect the bases of eigenfunctions and scar functions. This assumption is justified, in our opinion, because eigenfunctions and scar functions are of different nature. While a scar function is localized in the vicinity of a PO, an eigenfunction fills the surface of constant energy according to Shnirelman [4]. With this in mind, we expect that all the coefficients in equation (11) are *a priori* equiproblable, while the knowledge of one of them should not provide any information about the others[12]. Another justification, without invoking the Shnirelman theorem, is based on some knowledge of the interaction between scar functions. According to [13], the quantity $\rho_E H_{\gamma,\delta}$, where $H_{\gamma,\delta}$ is a typical non-diagonal Hamiltonian matrix element between scar functions, tends to infinity in the SL; that is, perturbation theory cannot be applied at all. Moreover, the precise value of each matrix element depends, in a very sensitive way, on classical invariants related to pairs of orbits; we might think about uncorrelated elements. So, it is reasonable to assume that after diagonalization, the amplitudes $\langle\varphi_\mu|\phi_\gamma\rangle$ are described by independent Gaussian distributions with zero means [15].

3. The scar phenomenon

In the following, we will compare the intensities of an eigenfunction on the basis of scar functions, with those resulting from a universal description. According to appendix A, we

[12] The opposite situation appears when two bases are connected by the perturbation theory, where a random wave hypothesis evidently does not work. If the elements of one basis are related to quantized tori, the elements of the perturbed basis are also related to tori (the perturbed ones). If the elements of one basis consist of chaotic eigenfunctions, the perturbed basis also contains chaotic eigenfunctions associated with the perturbed Hamiltonian.

need N_u universal waves for representing φ_μ, all of them with mean energy E_μ.[13] In this case, intensities are only characterized by their heights; therefore, the probability density of an individual intensity results

$$p_u(I) = \frac{\chi(I/\overline{I})}{\overline{I}}$$

where $\overline{I} = 1/N_u$ is the mean value of the intensities. Later on, the set of universal intensities of φ_μ is described by N_u independent random variables with common probability density $p_u(I)$. Using again the method of [16], the mean value of the greatest intensity results

$$I^{(u)} = \frac{2\ln N_u}{\beta N_u}[1 + \mathrm{O}(1/\ln N_u)]. \tag{14}$$

Moreover, relative fluctuations go to zero with $1/|\ln\hbar|$.

Let I_γ be a so strong intensity of φ_μ that it cannot be predicted by a model of random universal waves; that is, $I^{(u)} < I_\gamma$. This condition reveals that γ plays a distinctive role in the structure of φ_μ; according to Heller [6, 17], we say that φ_μ is scarred by γ. In what follows, we study how many intensities satisfy this condition, and more importantly, whether the sum of these intensities provides a relevant contribution to φ_μ in the SL. To reduce the number of variables playing a role in the discussion, we note that $\beta n I^{(u)}$ converges to a classical invariant in the SL. Using equation (14), then (4) (with σ in place of σ_γ) and finally equations (13) and (A.1), it results

$$\beta n I^{(u)} \simeq n\frac{2\ln N_u}{N_u} \simeq \sigma\rho_E\frac{2\ln N_u}{N_u} \simeq \eta$$

where

$$\eta \equiv \frac{(f-1)\left(\sum\lambda_i^2\right)^{1/2}}{h_T}. \tag{15}$$

Hence, we can rewrite the previous condition as

$$\eta < \beta n I_\gamma \quad \text{‘scarring condition’}. \tag{16}$$

We emphasize that $\beta n I^{(u)} = \mathrm{O}(\hbar^0)$ because two effects of different nature cancel each other. First, the factor $\ln N_u$ (which is $\mathrm{O}(|\ln\hbar|)$) is a direct consequence of the random wave hypothesis. Second, the factor n/N_u is of the order $1/|\ln\hbar|$ because the basis of scar functions incorporates the dynamics up to the Ehrenfest time. Let us go to analyse the consequences of this cancellation.

Using equations (7) and (9), the scarring condition takes the form

$$\eta = \mathrm{O}(\hbar^0) < \beta n x_j \simeq \beta n \overline{x}_j \sim \ln(n/j). \tag{17}$$

Note that the last relation is valid for $j \ll n$; in the other case, we have to replace the logarithmic function, by some other function of the same argument n/j. Then, fixing j, we see that x_j always satisfies the scarring condition in the SL because $\ln(n/j) \sim |\ln\hbar|$. Moreover, x_j is very strong in comparison with $I^{(u)}$ ($x_j/I^{(u)} \sim |\ln\hbar|$). On the other hand, a lot of intensities verify this condition. The number $n_{\rm scar}$ of intensities satisfying the scarring condition is of the order n, because an argument $n/n_{\rm scar}$ (in equation (17)) must be of the order $\hbar^0$.[14] Therefore, taking into account that the sum of all the intensities (where $\sum x_j = 1$)

[13] Mean energies of scar functions are the corresponding BS energies; however, in the case of universal waves such dynamical restriction does not exist.

[14] For $n_{\rm scar}/n \ll 1$, $n_{\rm scar}/n$ depends exponentially on η.

contains a number of order n of relevant terms, the first n_{scar} terms (the greatest ones) provide a finite contribution. For instance, when $n_{\text{scar}}/n \ll 1$, we can use equation (7) to obtain

$$\sum_{j=1}^{n_{\text{scar}}} x_j \sim \sum_{j=1}^{n_{\text{scar}}} \frac{\ln(n/j)}{n} \simeq \frac{n_{\text{scar}}}{n} \ln(n/n_{\text{scar}}).$$

We remark again that this result is a direct consequence of the use of scar functions and the random wave hypothesis; see comment after equation (16).

The previous analysis only uses the order of magnitude of the scarring condition; however, to obtain n_{scar}/n and the sum of scarred intensities in leading order, we have to consider equation (16) strictly (this relation includes two assumptions: the Gaussian behaviour proposed in equation (3) and equation (A.1)). The evaluation of these quantities for arbitrary η is cumbersome; nevertheless, starting with expansions around $\eta = 0$ and $\eta = \infty$, interpolation formulae are derived in [14]. For the $\beta = 1$ case, corresponding to the stadium billiard, they are given in terms of $\tilde{\eta} \equiv \sqrt{\pi/2}\eta$ by

$$\frac{n_{\text{scar}}}{n} \simeq \mathrm{e}^{-\tilde{\eta}} \sqrt{\frac{8 \ln(1 + 1/\tilde{\eta}\delta)}{(1 + 4\tilde{\eta}/\delta)}} \tag{18}$$

where $\delta = (9 + \sqrt{73})/2$, and

$$\sum_{j=1}^{n_{\text{scar}}} x_j \simeq \frac{2}{\sqrt{\pi}} \mathrm{e}^{-\tilde{\eta}} - \left(\frac{2}{\sqrt{\pi}} - 1\right) \mathrm{e}^{-2\tilde{\eta}}. \tag{19}$$

These formulae provide mean values of the quantities, while their dispersions are order of the $1/\sqrt{n}$. The accuracy of these expressions was checked with numerical experiments; note that the statistical model depends on the parameters η and n, and the previous equations correspond to the limit $n \to \infty$.

Now, we compare the results derived in the paper with the data obtained from a realistic system. Figure 1 shows the decomposition of an extremely excited eigenstate, φ, of the desymmetrized Bunimovich stadium billiard with radius 1 and area $1 + \pi/4$. Figure 1(a) displays the Husimi function of the state number 141 755, plotted in configuration space in [12]. We have evaluated its intensities on the basis of scar functions and also on the basis of plane waves; see [18] for the construction of scar functions in the stadium billiard. The highest scarred intensity is around five times greater than the highest intensity on the basis of plane waves, the ratio being of the order of $\ln n \simeq 3.8$. On the other hand, nine intensities of short POs satisfy the scarring condition, while equation (18) predicts $n_{\text{scar}} \simeq 8.5$. Moreover, the sum of scarred intensities is 0.38 in good agreement with equation (19), which predicts $\simeq$0.33.[15] Figure 1(b) shows the Husimi of the scarred contribution φ_{scar}, where

$$\varphi_{\text{scar}} \equiv \sum_{\text{scars}} \langle \phi_\gamma | \varphi \rangle \phi_\gamma$$

and figure 1(c) displays the Husimi of $\varphi - \varphi_{\text{scar}}$. We also plot the scarred intensities $|\langle \phi_\gamma | \varphi \rangle|^2$, arranged by height, and the theoretical curves $\overline{x_j} \pm \sigma_{x_j}$, observing a nice agreement[16]. At lower energies, we studied a series of 100 and 60 consecutive eigenfunctions around the states number 1500 and 5700, respectively; of course, we have rejected bouncing ball eigenstates. The

[15] We are unable to provide precise analytical values because, at present, there are only poor estimations for the topological entropy of the stadium billiard.

[16] We used precise expressions for $\overline{x_j}$ and σ_{x_j}, derived in [14].

selected energy regions satisfy, according to equation (18), $n_{\text{scar}} = 1$ and 2, respectively. Then, we evaluated the highest intensities on the basis of scar functions of short POs, obtaining the following main results: $\langle x_1 \rangle = 0.335 \pm 0.009$ in the first region, and $\langle x_1 + x_2 \rangle = 0.326 \pm 0.008$ in the second one[17]. Here, $\langle\,\rangle$ indicates mean value in the corresponding series of eigenfunctions. A detailed analysis of the data will be given elsewhere, but it is a clear remarkable agreement with the theory.

4. Conclusions

In conclusion, scar functions define a distinctive basis for the representation of chaotic eigenfunctions. In this representation, a generic chaotic eigenfunction is scarred by a particular set of short POs; this set characterizes the state. The number of elements in this set is given by equation (18), while the sum of their intensities survives the SL (see equation (19)). The scarred intensities have mean values and dispersions given by equations (7) and (8), respectively. Moreover, relative fluctuations being null in the SL (see equation (9)), mean values of the intensities provide a detailed description of chaotic eigenfunctions in this limit. With respect to the influence of a particular PO in a given energy range, equation (10) imposes a necessary condition of the Bohr–Sommerfeld quantization type for scarring.

These unexpected results are a consequence of the fact that scar functions incorporate the dynamics up to the Ehrenfest time. In contrast, by using wavefunctions of width order $\sqrt{\hbar}$ around the POs, for instance the so-called vacuum states in [12], the situation is completely different. The number of required elements on the basis of vacuum states is of the order $\hbar^{1-f}$, like it is for universal bases. So, the condition $I^{(u)} < I_\gamma$ is verified only eventually, with greater probability by those POs with smaller Lyapunov exponents. For this reason, by using vacuum states, the scar phenomenon results exceptionally in the SL. At low energies, it appears clearly in favourable situations [19], while there are serious doubts about its existence in other ones [20].

Finally, we would like to state some remarks.

(a) Using a random wave hypothesis, we conclude that the sum of scarred intensities of a generic chaotic eigenfunction survives the SL. Moreover, by accepting equations (3) and (A.1), we obtain its limiting value.

(b) Numerical results in the stadium billiard demonstrate that, even though the statistical model is derived in the extreme semiclassical limit, the obtained conclusions are of remarkable validity at low energies.

(c) Relative fluctuations of the intensities tend to zero in the SL (see equation (9)); however, it should be clear that at low energies they are very important. For this reason, the phenomenon of localization on short POs can be very strong for some eigenfunctions at low energies.

(d) A scar function is not able to support an eigenfunction in the SL because $\rho_E \sigma_\gamma \gg 1$, where $\rho_E = \mathrm{O}(\hbar^{-f})$ and $\sigma_\gamma = \mathrm{O}(\hbar/|\ln \hbar|)$. Actually, the right strong localization condition should be $\tilde{\rho}_E \sigma_\gamma < 1$, where $\tilde{\rho}_E$ is the density of energy levels. Suppose, there is a chaotic system with a so strong spectral degeneracy that $\rho_E \gg \tilde{\rho}_E = \mathrm{O}(|\ln \hbar|/\hbar)$; in this case, a scar function is able to satisfy the strong localization condition. This non-generic situation appears exceptionally in cat maps [21].

(e) We can derive general conclusions about the phenomenon of localization on short POs, because it is governed by the classical invariant η. For instance, taking into account that the sum of scarred intensities is a decreasing function of η (see equation (19)), while η

[17] The error of $\langle x \rangle$ is given by $\sigma_x/\sqrt{N_{\text{eig}}}$, where N_{eig} is the number of eigenfunctions in the corresponding series.

increases with f,[18] we conclude that this localization is stronger in systems with few degrees of freedom.

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Appendix A

We will discuss how many independent universal waves participate in the description of chaotic eigenfunctions; this point is relevant in order to obtain a quantitative criterion for scarring. For us, universal waves are suitable wavefunctions without information about the structure of POs of the system, for instance plane waves are universal waves in billiards.

In order to solve the question, we have to find a characteristic time, describing the transition from universal to non-universal evolutions. At first, there are several possibilities. For instance, we can use the period T_1 of the shortest PO, because it is impossible to extract information about the structure of POs from evolutions shorter than T_1. Nevertheless, T_1 only provides information about a very limited region of phase space, while the required time should be the result of a deep knowledge of the system. For this reason, we propose the inverse of the topological entropy for this time. That is, a classical invariant of the order of the shortest POs, which further defines the proliferation of POs as a function of the period.

Then, taking into account that an eigenfunction is defined after an evolution for a time $T_{\rm H}$, by considering this evolution in stages of time h_T^{-1}, where each one establishes a universal wave, the number of independent universal waves results

$$N_u \simeq T_{\rm H} h_T = 2\pi\hbar\rho_E h_T = {\rm O}(\hbar^{1-f}). \tag{A.1}$$

We will try to justify the plausibility of this equation, and clarify the meaning of *independent* universal waves. Let us consider for example, the Sinai billiard; a square of side unity with an interior circle of radius R. From the classical point of view, it is possible to analyse the motion with Birkhoff coordinates on the boundary of the square, or equivalently on the boundary of circle. The corresponding surface of sections provides the same information, but their areas are different $A = 8p$ for the square, while $A = 4\pi Rp$ for the circle (p is the modulus of the momentum of the particle). Such a discrepancy is very relevant at the quantum level because for the evaluation of eigenfunctions, we need a basis of $N \simeq A/2\pi\hbar$ wavefunctions associated with the section. For instance, by considering the surface of section related to the square, we use the following suitable set of wavefunctions with defined wave number k:[19]

$$\psi_l(r,\theta) = [a_l J_l(kr) + b_l Y_l(kr)]\,{\rm e}^{{\rm i}l\theta} \qquad \text{for} \quad l = 0, \pm 1, \ldots, \pm L \simeq 2k/\pi.$$

J_l and Y_l are Bessel and Neumann functions of order l, respectively, and the coefficients a_l and b_l are selected in order to satisfy boundary conditions on the circle. Using the same idea for the other surface of section, we have to find wavefunctions, with defined wave number, satisfying boundary conditions on the square; the construction of such wavefunctions is explained in [22].

[18] Assuming that $h_T \sim \sum \lambda_i$, η results an increasing function of f independently of the distribution of Lyapunov exponents.

[19] Polar coordinates take the origin at the centre of the circle.

For $R = \mathrm{O}(1)$ the two bases are comparable because $N \simeq 4k/\pi$ for the square, while $N \simeq 2Rk$ for the circle. However, it is evident that as $R \to 0$ the first basis turns into very inefficient with respect to the second optimal one. In such a situation, the set $\{\psi_l\}$ must be strongly correlated and consequently, the representation $\varphi = \sum \langle \psi_l|\varphi\rangle \psi_l$ of eigenfunctions cannot be naively described by a random wave.

Finally, equation (A.1) predicts

$$N_u \simeq L_{\mathrm{H}} \tilde{h}_T = (1 - \pi R^2) k \tilde{h}_T$$

where L_{H} is the Heisenberg length, and $\tilde{h}_T$ is the topological entropy per unit length. With respect to $\tilde{h}_T$, it does not exist to our knowledge an evaluation of this quantity as a function of R. However, the rough estimate $\tilde{h}_T \sim \tilde{\lambda} \simeq 4R \ln(1/R)$, where $\tilde{\lambda}$ is the Lyapunov exponent per unit length, suggests that equation (A.1) provides the main behaviour of the optimal basis.

In conclusion, equation (A.1) attempts to obtain the minimum number of universal waves required for the representation of chaotic eigenfunctions. In this way, we can speak of independent waves and therefore, a random wave description results reasonable.

Appendix B

This appendix does not provide relevant results for the development of the paper; however, it helps to clarify some properties of the set of short POs used to construct the scar function basis. In particular, we establish the short PO condition.

The first requirement of the set of short POs is that the density of BS energies equalizes ρ_E [10]. $\rho_\gamma \simeq T_\gamma/2\pi\hbar$ being the density of BS energies of γ, the condition $\sum_\gamma \rho_\gamma \simeq \rho_E$ immediately results

$$T_1 + T_2 + \cdots + T_{N_{\mathrm{po}}} \simeq T_{\mathrm{H}} \tag{B.1}$$

where T_1 is the period of the shortest PO, T_2 is the period of the next one and so on. Here, we arrange the periods in order of magnitude, and select the first N_{po} POs. This criterion appears reasonable as a first approach; however, we will see below that asymptotic arguments provide a better one.

The period $T_{N_{\mathrm{po}}}$ of the longer short PO is evaluated as follows:

$$T_1 + \cdots + T_{N_{\mathrm{po}}} \simeq \int_{T_1}^{T_{N_{\mathrm{po}}}} T\rho(T)\,\mathrm{d}T \sim \int_{T_1}^{T_{N_{\mathrm{po}}}} \mathrm{e}^{h_T T}\,\mathrm{d}T \simeq \mathrm{e}^{h_T T_{N_{\mathrm{po}}}}$$

where $\rho(T) \sim \mathrm{e}^{h_T T}/T$ is the density of POs with period T. Therefore, taking the logarithm in equation (B.1), it results

$$h_T T_{N_{\mathrm{po}}} \simeq (f-1)|\ln \hbar|. \tag{B.2}$$

A direct application of the last relation provides the following short PO condition:

$$T_\gamma < T_{N_{\mathrm{po}}} \simeq (f-1)|\ln \hbar|/h_T \tag{B.3}$$

where the relative error on the rhs is of the order $1/|\ln \hbar|$. To reduce this error, we note that $\sigma/\hbar \sim 1/|\ln \hbar|$ (see equation (13)). Then, we change slightly the condition as follows:

$$T_\gamma \sigma_\gamma/\hbar < T_{N_{\mathrm{po}}} \sigma/\hbar.$$

Hence, with the help of equations (13), (15) and (B.2), we see that the rhs takes a finite limiting value

$$T_\gamma \sigma_\gamma/\hbar < \pi\eta \qquad \text{'short PO condition'}. \tag{B.4}$$

The obvious advantage of this condition, with respect to equation (B.3), is that all quantities can be obtained, in principle, with high accuracy.

As a bonus, it is worth emphasizing that by using the short PO condition proposed in equation (B.4), all isolated POs of the stadium billiard are considered on the same footing, including the whispering gallery family and the POs near the bouncing ball region. For instance, taking into account that the whispering gallery family accumulates at a finite period $T_{\mathrm{w.g}}$, it is obvious that the condition $T_\gamma < T_{N_{\mathrm{po}}}$ incorporates this infinite set of POs when $T_{\mathrm{w.g}}$ satisfies $T_{\mathrm{w.g}} < T_{N_{\mathrm{po}}}$. In contrast, as σ_γ increases with λ_γ, and λ_γ increases logarithmically with the number l of bounces with the circle, it is clear that equation (B.4) introduces a cut-off to the admitted POs of this family. As $\lambda_\gamma T_{\mathrm{w.g}} \simeq 2\ln l$ [23], it results from (B.4) that the number of POs of this family is $\mathrm{O}(l) = \mathrm{O}(\hbar^{-\eta/2})$. Assuming that $\eta \simeq 1$ for the stadium billiard, the number of POs agrees with the number of states in the region. In Birkhoff coordinates, the whispering gallery region is of the order $\hbar^{1/2}$; so the number of states in this region is of the order $\hbar^{-1/2}$.

For orbits near the bouncing ball region, we have another situation. Bouncing ball eigenstates cover a band of width $\mathrm{O}(\hbar^{1/2})$ around $p = 0$; see the white band in figure 1(*a*). Near this band but in the chaotic region, POs have very large periods of the order $\hbar^{-1/2}$; so, these orbits do not satisfy equation (B.3). This problem is solved by using equation (B.4) because $\lambda_\gamma T_\gamma \simeq \ln l \simeq |\ln\hbar|/2$, where l is the number of bounces with the straight line.

We would like to note that the set of short POs should be distributed in phase space according to the classical ergodic measure. This implicit assumption is related to the way in which eigenfunctions fill phase space, because the most economic basis for representing eigenfunctions should be in terms of structures distributed according to the same rule. In this respect, equation (B.4) has demonstrated to be very efficient for avoiding serious pathologies associated with nonhyperbolic systems. For this reason, we hope that this condition yields a suitable PO covering of phase space for very general chaotic systems.

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